

Group Theory – Post 2018 Mix – Mark Scheme

1.

(i)	$X = \sqrt[3]{\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}} \Rightarrow X^2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, X^3 = \sqrt[3]{\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}},$ $X^4 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, X^5 = \sqrt[3]{\begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}}, X^6 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Note that $X^7 = -X$ etc. so that none of X^7 to X^{11} is equal to I $\Rightarrow X^{12} = I$ and so $n(M) = 12$ because the powers of X now cycle every 12^{th} power but no sooner	M1 A1 A1 E1 [4]	1.1a 1.1 2.2a 2.4	Attempt at powers of X (possibly BC) Correct up to X^6 (NB – not <i>all</i> needed) Candidates may calculate each of these BC Correctly deduced Clear statement needed to justify $n(M) = 12$
(ii)	The set is closed (each element is one of the 12 powers of X); matrix multiplication is known to be associative; the multiplicative identity I is in M; and the inverse of X^k is X^{12-k} for all suitable $k \dots$ so we have a group G The group is cyclic since it is generated by X (or X^3 or X^7 or X^{11}) oe	B1 B1 B1 B1 E1 [5]	2.1 1.1 1.1a 1.1 2.4	One for each group axiom Inverses may be explicitly paired-up Only one generator needs to be mentioned here
(iii)	Proper subgroups of G are $H_2 = \{X^6, I\}$, $H_3 = \{X^4, X^8, I\}$, $H_4 = \{X^3, X^6, X^9, I\}$, $H_6 = \{X^2, X^4, X^6, X^8, X^{10}, I\}$ The above subgroups are those generated as follows: $H_2 = \langle X^6 \rangle$, $H_3 = \langle X^4 \rangle = \langle X^8 \rangle$, $H_4 = \langle X^3 \rangle = \langle X^9 \rangle$ and $H_6 = \langle X^2 \rangle = \langle X^{10} \rangle$ and are thus unique Mention that <i>Lagrange's Theorem</i> states that $H \mid G$ so there are no subgroups of orders 5, 7, 8, 9, 10 or 11 OR For listing the subgroups (as above) The elements have orders $\{12, 6, 4, 3, 12, 2, 12, 3, 4, 6, 12, 1\}$ and one can argue from the orders of elements that (e.g.) : a subgroup of order 2 must contain I and an element of order 2 $\Rightarrow H_2$ uniquely; a subgroup of order 3 must contain I and an element of order 3 $\Rightarrow H_3$ uniquely, since there are only two such elements and they are an inverse-pair; similarly for H_6 Then as above for no subgroups of orders 5, 7, 8, 9, 10 or 11	B1 B1 B1 E1 [5] B1 B1 B1 E1 B1	3.1a 1.1 3.1a 3.1a 2.2a	Any two correct; All four correct (+ no extras – ignore inclusion of $\{I\}$ and/or G) Condone omission to mention that any subgroup containing X^5 or X^7 or X^{11} is G (since they are generators)

2.

(i)	If $f^2(x) = x$ or $f^3(x) = x$ then it will generate a group of order 2 or 3.	B1 B1 [2]	2.3 2.2a	
(ii)	Since $\det(\text{product}) = \text{product of dets}$ we need $\det(M^6) = (\det M)^6 = \det(I) = 1 \Rightarrow \det(M) = \pm 1$	E1 E1 [2]	3.1a 2.1	
(iii)	Want $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \frac{2\pi}{6} & -\sin \frac{2\pi}{6} \\ \sin \frac{2\pi}{6} & \cos \frac{2\pi}{6} \end{pmatrix}$	B1 [1]	3.4	i.e. a generator of the group of rotations through multiples of $\frac{\pi}{3}$
(iv)	(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ gives $f(x) = \frac{\frac{1}{2}x - \frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}x + \frac{1}{2}}$ or $\frac{x - \sqrt{3}}{x\sqrt{3} + 1}$	B1 [1]	3.5a	
	(b) $M^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ gives $f^3(x) = \frac{-x+0}{0x-1} = x$, so f not of order 6	B1 B1 [2]	3.5b 3.1a	* BC Must state the conclusion and not stop with the working
	(c) However, taking $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \frac{\pi}{6} & -\sin \frac{\pi}{6} \\ \sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{pmatrix}$ gives $f(x) = \frac{x\sqrt{3}-1}{x+\sqrt{3}}$ for which $f^2(x) = \frac{x-\sqrt{3}}{x\sqrt{3}+1}$ and $f^3(x) = \frac{-1}{x} \Rightarrow f^6(x) = x$ Must state, or have shown, that both $f^2(x)$ and $f^3(x) \neq x$	M1 A1 M1 A1 [4]	3.5c 3.2a 2.4	Another rotation matrix A correct suggestion Attempt at powers of f up to f^6

3.

(a)	(i)	Since $a \neq 0$ and $c \neq 0$, $ac \neq 0 \Rightarrow (ac, b + ad) \in L$ so L is closed under $*$	B1 [1]	2.4	
	(ii)	$[(a, b) \diamond (c, d)] \diamond (e, f) = (ac, b + ad) \diamond (e, f)$ $= (ace, b + ad + acf)$ $(a, b) \diamond [(c, d) \diamond (e, f)] = (a, b) \diamond (ce, d + cf)$ $= (ace, b + a(d + cf))$ Since $a(d + cf) = ad + acf$ the two are equal, we have associativity	M1 A1 M1 A1 [4]	1.1a 1.1 2.1 2.4	Attempt at $(x \diamond y) \diamond z$ or $x \diamond (y \diamond z)$ for elements x, y, z The other attempted Must state that the two are equal
	(iii)	$(a, b) \diamond (c, d) = (ac, b + ad) = (a, b)$ e.g. $\Rightarrow c = 1, d = 0$ i.e. $(1, 0)$ is the identity	M1 A1 [2]	1.1a 1.1	Setting product equal to either component
	(iv)	$(a, b) \diamond (c, d) = (ac, b + ad) = (1, 0)$ gives $c = \frac{1}{a}$ and $d = -\frac{b}{a}$ so that $(a, b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a}\right)$ which exists and is in L since $a \neq 0$ and $\frac{1}{a} \neq 0$	M1 A1 E1 [3]	1.1a 1.1 2.4	Must note here that the inverse is in L
	(b)	$(a, b) * (a, b) = (1, 0)$ i.e. $(a^2, b(1+a)) = (1, 0)$ For this not to be the identity, $a = -1$ and b arbitrary So the set $\{(1, 0), (-1, b)\}$ for any real b will form a subgroup of order 2	M1 A1 [2]	3.1a 2.2a	Looking for any self-inverse element Must choose a value for b or note that any will do

4.

(i)	(a)	$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$ i.e. $p = ac - bd$ and $q = ad + bc$	M1	1.1	May be clear from the product matrix	
		A1 [2]	1.1			
(i)	(b)	Suppose that both $p = 0$ and $q = 0$, so that $ac - bd = 0$ and $bc + ad = 0$	B1	1.1	Explicit assumption that both are zero Condition that a and b not both zero and c and d not both zero need not be explicitly stated	OR B1 Suppose that both $p = 0$ and $q = 0$ so that $\det \begin{pmatrix} p & -q \\ q & p \end{pmatrix} = p^2 + q^2 = 0$ M1 Then $\det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = 0$ A1 so that $c^2 + d^2 = 0$ and/or $a^2 + b^2 = 0$ A1 since neither $\{c \text{ and } d\}$ or $\{a \text{ and } b\}$ can both be zero... E1 This contradicts the condition that a, b and c, d are not both zero, so the assumption that p and q are both zero is false.
		Then $acd - bd^2 = 0$ and $acd + bc^2 = 0$ Subtracting $\Rightarrow b(c^2 + d^2) = 0$ Since c and d are not both zero $c^2 + d^2 \neq 0 \Rightarrow b = 0$ Similarly, $ac - bd = 0$ and $bc + ad = 0 \Rightarrow a(c^2 + d^2) = 0$ Since c and d are not both zero $c^2 + d^2 \neq 0 \Rightarrow a = 0$ This contradicts the condition that a, b are not both zero, so the assumption that p and q are both zero is false.	M1	3.1a	e.g. treating as simultaneous equations (either a, b or c, d)	
		A1	2.2a	One case correctly concluded		
		A1	2.2a	Second case correctly concluded		
		E1	2.4	Contradiction clearly explained		
		[5]				

(ii)		Closure follows from part (i) (Associativity is given) Identity is when $a = 1, b = 0$ or noting $\begin{pmatrix} 1 & (-)0 \\ 0 & 1 \end{pmatrix} \in X$ Inverse of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is $\frac{1}{a^2 + b^2} \begin{pmatrix} a & -(-b) \\ -b & a \end{pmatrix} \in X$ since $a^2 + b^2 \neq 0$ (as a, b not both zero)	B1	1.1	To earn this, they must show appropriate form for $\mathbf{A}^{-1}$ and note why it exists	
		B1	1.1			
		B1	2.1			
		E1 [4]	2.3			
(iii)		The matrix with $a = \cos(\frac{2}{17}\pi), b = \sin(\frac{2}{17}\pi)$ generates ... the subgroup of rotations about O (through multiples of $\frac{2}{17}\pi$)	B1	3.1a	Suitable values of a and b	
		B1	2.4	Description of the subgroup		
			[2]			

5.

(a)	$11^2 = 121 \equiv 41 \pmod{80}$ so $41 \in S$ $11 \times 41 = 451 \equiv 51 \pmod{80}$ so $51 \in S$ $41^2 \equiv 1 \pmod{80}$ so $1 \in S$	3.1a 2.5 2.2a	B1 B1 B1	B1 for each other element found (possibly tabulated)																								
	<table border="1"> <tr> <td>$\times_{80}$</td><td>1</td><td>11</td><td>41</td><td>51</td></tr> <tr> <td>1</td><td>1</td><td>11</td><td>41</td><td>51</td></tr> <tr> <td>11</td><td>11</td><td>41</td><td>51</td><td>1</td></tr> <tr> <td>41</td><td>41</td><td>51</td><td>1</td><td>11</td></tr> <tr> <td>51</td><td>51</td><td>1</td><td>11</td><td>41</td></tr> </table>	$\times_{80}$	1	11	41	51	1	1	11	41	51	11	11	41	51	1	41	41	51	1	11	51	51	1	11	41	1.1a 1.1	M1 A1
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51	51	1	11	41																								
[5]																												
(b)	Choice of a suitable method, such as Listing orders of elements: 1, 4, 2, 4 in G and 1, 2, 2, 2 in H	2.1	M1	MUST be for two groups of order 4																								
	OR Stating that G is the cyclic group of order 4 while H is the Klein-4 group OR Stating that G has an element of order 4 while H has all (non-identity) elements of order 2	2.4	A1	Using knowledge of small-order groups: M1 for clear indication that there are only two groups of order 4, A1 for the details																								
[2]																												
(c)	i G has proper subgroup $\{1, 41\}$ only H has proper subgroups $\{1, 9\}$, $\{1, 31\}$ and $\{1, 39\}$	3.1a 1.2 2.2a	B1 M1 A1 [3]	Ignore statement of $\{1\}$ and group throughout Listing any two correct subgroups of order 2 All three and no extras																								
	ii G and H must have different structures since they have differing numbers of (proper) subgroups	2.4	B1	(Must have different numbers of subgroups in (c)(i))																								
[11]																												

6.

(a)		<table><tr><td></td><td>a</td><td>b</td><td>c</td><td>d</td></tr><tr><td>a</td><td>b</td><td>d</td><td>a</td><td>c</td></tr><tr><td>b</td><td>c</td><td>a</td><td>d</td><td>b</td></tr><tr><td>c</td><td>a</td><td>b</td><td>c</td><td>d</td></tr><tr><td>d</td><td>d</td><td>c</td><td>b</td><td>a</td></tr></table>		a	b	c	d	a	b	d	a	c	b	c	a	d	b	c	a	b	c	d	d	d	c	b	a	<table><tr><td>Column 1</td></tr><tr><td>Column 3</td></tr><tr><td>Any 3rd R or C</td></tr><tr><td>All correct</td></tr></table>	Column 1	Column 3	Any 3 rd R or C	All correct	<table><tr><td>B1</td><td>3.1a</td></tr><tr><td>B1</td><td>1.1</td></tr><tr><td>B1</td><td>2.2a</td></tr><tr><td>B1</td><td>1.1</td></tr><tr><td>[4]</td><td></td></tr></table>	B1	3.1a	B1	1.1	B1	2.2a	B1	1.1	[4]		
		a	b	c	d																																							
	a	b	d	a	c																																							
	b	c	a	d	b																																							
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B1	1.1																																											
B1	2.2a																																											
B1	1.1																																											
[4]																																												
(b)	No – since c is the “left identity” but there is no “right identity” Or there is no identity column Or $(ad)b = cb = b$ <u>but</u> $a(db) = ac = a$ so op. not associative	<table><tr><td>B1</td><td>2.4</td></tr><tr><td>[1]</td><td></td></tr></table>	B1	2.4	[1]		2.4	Any valid reason (vague or unsupported claims are not acceptable)																																				
B1	2.4																																											
[1]																																												

7.

(a)	2, 3, 4 and 6 because these are the factors of 12 (by Lagrange’s Theorem)	B1 B1 [2]	1.1 2.4	Ignore 1 or 12 if they appear
(b)	$\{3, 9, 27\}$	B1 [1]	1.1	
(c)	27	B1 [1]	1.1	
(d)	$18, 18^2 = 324 \equiv 12 \pmod{39}, (18^3 = 21) \dots$ $18^4 = 12^2 = 27 \pmod{39}$ so order 4	M1 A1 [2]	1.1 2.1	For method of searching by finding powers
(e)	$3 \equiv 3, 42, 81, \dots$ First square-root of 3 is 9 Second square-root of 3 is $-9 \equiv 30$	M1 A1 A1 [3]	3.1a 2.2a 3.2a	Method for searching for squares $\equiv 3 \pmod{39}$ Cannot just state -9
(f)	Identity 27 in G does not map to identity 1 in H	B1 [1]	2.4	Or any clear equivalent statement

8.

(a)	$\{e, a, a^2\}$	B1	1.1	Ignore inclusion of $\{e\}$, G
	$\{e, b\} \quad \{e, ab\} \quad \{e, a^2b\}$	B1 B1 B1 [4]	1.1 1.1 1.1 1.1	
(b)	G is isomorphic to the symmetric group of three elements i.e. D_3 (the symmetries of the Δ) or S_3 (the permutations of 3 symbols)	B1 [1]	2.5	Accept “the non-cyclic group of order 6”

9.

(a)	Closed ... YES since the 'product' of 2 complex numbers is a third	B1 [1]	2.4	'Yes' with valid reason
(b)	$[(a + ib) \diamond (c + id)] \diamond (e + if) = [ac + i(b + ad)] \diamond (e + if)$ $= ace + i(b + ad + acf)$ $(a + ib) \diamond [(c + id) \diamond (e + if)] = (a + ib) \diamond [ce + i(d + cf)]$ $= ace + i(b + a[d + cf]) = \dots$	M1 A1 M1 A1 [4]	1.2 1.1 2.1 1.1	Attempt at first of $(pq)r$ or $p(qr)$, correct to here Attempt at the other, correct to here Must be convincingly shown equal to first answer
(c)	$1 + i0$ is the identity since $(1 + i0) \diamond (a + ib) = 1 \cdot a + i(0 + 1b) = (a + ib) \diamond (1 + i0)$	B1 B1 [2]	2.2a 2.4	Demonstrated (condone one-side only)
(d)	$(a + ib) \diamond (c + id) = 1 \Rightarrow ac + i(b + ad) = 1$ $\Rightarrow c = \frac{1}{a}$ and $d = \frac{-b}{a}$ S is the set of complex numbers with non-zero real part (The 'exclusions' are those complex numbers with zero real part; i.e. the strictly imaginary numbers)	M1 A1 B1 [3]	3.1a 2.2a 3.2a	Test for an inverse of a general element Both components of inverse element found Explained

10.

a	Inverses pair up (even number) and there is one identity element Since total is even, there must be an odd number of self-inverse elements	M1 A1 [2]	2.1 2.2a	Parity argument involving inverse pairs
b	Consider xy and its inverse, also xy . Now $xy = (xy)^{-1}$ $= y^{-1}x^{-1} = yx \ (\Rightarrow \text{abelian})$	M1 A1 [2]	3.1a 1.1	Must be stated or derived NB No need to include statements about the identity
c	By considering the product of any two of elements a, b, c gives one of d, e of order 3	M1 A1 [2]	2.1 2.2a	It must be noted that these elements are self-inverse (or of order 2) Correct example from the Cayley table must be given and the contradiction concluded
d	Assume that all (non-identity) elements of G are self-inverse/have order 2 Then H is a group ($\cong$ the Klein group of order 4) and hence a subgroup of G (Lagrange's theorem states that) $o(H) \mid o(G)$ But $4 \nmid (4n + 2)$ giving the required $\Rightarrow \Leftarrow$	B1 B1* M1 A1dep [4]	2.1 2.2a 2.4 1.1	Longer proofs that H is a group may involve $xy = yx$ (by part b) so that H is closed, etc. Consideration of Lagrange's theorem

11.

i	<table border="1"> <tr> <td>$\times_{32}$</td><td>1</td><td>9</td><td>17</td><td>25</td></tr> <tr> <td>1</td><td>1</td><td>9</td><td>17</td><td>25</td></tr> <tr> <td>9</td><td>9</td><td>17</td><td>25</td><td>1</td></tr> <tr> <td>17</td><td>17</td><td>25</td><td>1</td><td>9</td></tr> <tr> <td>25</td><td>25</td><td>1</td><td>9</td><td>17</td></tr> </table>	$\times_{32}$	1	9	17	25	1	1	9	17	25	9	9	17	25	1	17	17	25	1	9	25	25	1	9	17	B1 B1 [2]	1.1 1.1	Any two bold entries (in shaded squares) correct All bold entries (in shaded squares) correct
$\times_{32}$	1	9	17	25																									
1	1	9	17	25																									
9	9	17	25	1																									
17	17	25	1	9																									
25	25	1	9	17																									
ii	$G \cong C_4$ Since either G has a generator (9 or 25) or from the main diagonal, only 2 of G 's elements are self-inverse, so $G \not\cong K_4$	B1 B1 [2]	2.2a 2.4	Allow statement that G is cyclic Either generator noted Or statement that only 1 non-identity element is self-inverse																									

12.

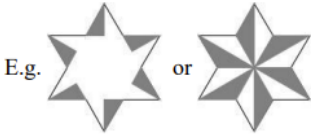
(a)	Require b s.t. $a \oplus b = a$ i.e. $\frac{a+b}{ab+1} = a$ $\Leftrightarrow b = 0$	M1 A1 [2]	1.1 2.2a	Including attempt to solve (e.g. by multiplying across), possibly implied by correct answer ("since $a^2 \neq 1$ " need not be mentioned) SC1 verifying that 0 is the identity element
(b)	Require b s.t. $a \oplus b = 0$ $\Rightarrow b = -a$ i.e. $x^{-1} = -x$	M1 A1 [2]	2.1 1.1	(Or for x) (NB 0 is 'self-inverse')
(c)	$(a \oplus b) \oplus c = \frac{a+b}{ab+1} \oplus c$ attempted $a \oplus (b \oplus c) = a \oplus \frac{b+c}{bc+1}$ attempted Both shown equal to $\frac{a+b+c+abc}{ab+ac+bc+1}$	M1 M1 A1 [3]	1.1 1.1 1.1	OR noting symmetry in (a, b, c) of answer Or any equivalently correct equal forms
(d)	The set C is not closed under $\oplus$ since $a \oplus b$ undefined for any $(a, b) = \left(x, -\frac{1}{x}\right)$, $x \neq 0, \pm 1$	B1 [1]	2.3	Closure shown not to apply (i.e. with example, general or specific)
(e)	"Subgroup" of order 3 must be of the form $\{0, a, -a\}$ Require $a \oplus a = -a$ or $a \oplus a \oplus a = 0$ $a = \pm i\sqrt{3}$ from solving cubic equation ($a \neq 0$)	B1 M1 A1 [3]	2.1 3.1a 3.2a	Must be made clear from the working; may state $\{0, a, a \oplus a\}$ initially, instead oe

13.

6(i)	$(a*b)*c = (a+b+ab)*c = a+b+ab+c+(a+b+ab)c$	M1	2.1														
	$a*(b*c) = a*(b+c+bc) = a+b+c+bc+a(b+c+bc)$	M1	2.1														
	$\frac{a+b+ab+c+(a+b+ab)c}{= a+b+c+bc+ab+ac+abc}$ $= \frac{a+b+c+bc+a(b+c+bc)}{}$	A1	2.2a														
	so $(a*b)*c = a*(b*c)$ which means * is associative	A1	2.4														
		(4)															
(ii)(a)	$3^2 = 2 \quad 3^3 = 6 \quad 3^4 = 4 \quad 3^5 = 5 \quad 3^6 = 1$ or $5^2 = 4 \quad 5^3 = 6 \quad 5^4 = 2 \quad 5^5 = 3 \quad 5^6 = 1$	M1	2.1														
	Or special case for M1A0 if powers not shown: 3 has order 6 so generates the group																
	3 (or 5) has order 6 and so generates the group so G is cyclic	A1	2.4														
		(2)															
(b)	$\{1\}, H$	B1	1.1b														
	$\{1, 17\}$ or $\{1, 7, 13\}$	M1	1.1b														
	$\{1, 17\}$ and $\{1, 7, 13\}$ (and no others)	A1	1.1b														
		(3)															
(c)	<table><tr><td>G</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>H</td><td>1</td><td>7</td><td>5</td><td>13</td><td>11</td><td>17</td></tr></table>	G	1	2	3	4	5	6	H	1	7	5	13	11	17	M1	3.1a
	G	1	2	3	4	5	6										
	H	1	7	5	13	11	17										
	or	A1	1.1b														
<table><tr><td>G</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>H</td><td>1</td><td>13</td><td>11</td><td>7</td><td>5</td><td>17</td></tr></table>	G	1	2	3	4	5	6	H	1	13	11	7	5	17	A1	1.1b	
G	1	2	3	4	5	6											
H	1	13	11	7	5	17											
	(3)																

(12 marks)

14.

6(a)	A rotation ...	M1	1.1b
	... about the centre of the shape, through an angle 120° anticlockwise.	A1	1.1b
		(2)	
(b)	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$	B1	1.1b
	One of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$	M1	1.1b
	Two of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ & $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$	A1	1.1b
	All the above and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ and no extra symmetries given. Two line form required for this mark.	A1	2.5
		(4)	
(c)	G has order 6 so can have no subgroup of order 4 by Lagrange's Theorem.	B1	2.4
	There is no element of order 6 that generates the group.	B1	2.4
		(2)	
(d)	E.g.  or	M1 – breaks the reflections A1 – keeps all rotations and no reflections	M1 3.1a
			A1 1.1b
		(2)	

(10 marks)

15.

(a)	For $m, n \in \mathbb{Z}_0^+$ we have $m - n \in \mathbb{Z}$ (difference of integers is an integer) and so $ m - n \in \mathbb{Z}_0^+$, hence closed under $\star$.	B1	2.4
		(1)	
b)	For $m \in \mathbb{Z}_0^+$, $0 \star m = 0 - m = -m = m$ and $m \star 0 = m - 0 = m = m$ Hence 0 is an identity*.	Checks either side	M1 1.1b
		Checks both sides and makes conclusion.	A1* 2.1
		(2)	
c)	For $m \in \mathbb{Z}_0^+$, we need $ m - n = 0 \Rightarrow n = \dots$ or shows $ m - m = 0 = 0$	M1	2.2a
	As $ m - m = 0$ for all $m \in \mathbb{Z}_0^+$ each m is self-inverse.	A1	2.1
		(2)	
d)	Checks associativity – ie evaluates $m \star (n \star p)$ and $(m \star n) \star p$ with letter or numbers.	M1	1.2
	E.g. $1 \star (2 \star 3) = 1 \star 2 - 3 = 1 \star 1 = 0$ but $(1 \star 2) \star 3 = 1 - 2 \star 3 = 1 \star 3 = 1 - 3 = 2$	M1	3.1a
	$1 \star (2 \star 3) \neq (1 \star 2) \star 3$ hence not associative so not a group.	A1	2.4
		(3)	

(8 marks)

16.

1(a)	$\{e\} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	B1	1.1b
		(1)	
(b)	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$	B1	1.1b
		(1)	
(c)	Demonstrates that, for example: $[a \circ b] \circ c = \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \right] \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ $a \circ [b \circ c] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right]$ $= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	M1	2.1
	So $[a \circ b] \circ c = a \circ [b \circ c]$ or associative	A1	2.4
		(2)	
(d)	The order of the group is 24 or 4!	B1	1.1b
	4 is a factor of 24 or 4/24 therefore it is possible for a subgroup to have order 4.	B1ft	2.4
		(2)	

(6 marks)

17.

(a)	$x \bullet e = x \Rightarrow 3(x + e + 1) + 2xe = x$ $\Rightarrow 3e + 2xe = x - 3x - 3 \Rightarrow e = \dots$ Or Let $e = -1 \Rightarrow e \bullet y = 3(-1 + y + 1) + 2(-1)y = \dots$	M1	2.1
	$\Rightarrow e = -\frac{2x+3}{2x+3} \Rightarrow e = -1$ Or $e \bullet y = 3(-1 + y + 1) + 2(-1)y = y$ therefore $e = -1$	A1	2.2a
		(2)	
(b)	Let the inverse of x be y then $x \bullet y = e \Rightarrow 3(x + y + 1) + 2xy = -1$	M1	3.1a
	$\Rightarrow 3y + 2xy = -1 - 3x - 3 \Rightarrow y = \dots$	M1	2.1
	$\Rightarrow x^{-1} = -\frac{3x+4}{2x+3}$ oe such as $-\frac{3}{2} + \frac{1}{4x+6}$	A1	2.2a
		(3)	
(c)	E.g. for $x = -\frac{3}{2}$ the identity property would fail since we cannot divide by $2x + 3 = 0$ or shows $x^{-1} = -\frac{1}{0}$ so no inverse or can't divide by 0 Or $-\frac{3}{2} \bullet x = 3\left(-\frac{3}{2} + x + 1\right) - 3x = -\frac{3}{2}$ so the identity property fails as not unique as any x would be an identity for $-\frac{3}{2}$.	B1	2.4
		(1)	

(6 Marks)

18.

4(i)	If we assume $ab = ba$; as $a^2b = ba$ then $ab = a^2b$	M 1	2.1
	So $a^{-1}abb^{-1} = a^{-1}a^2bb^{-1}$	M 1	2.1
	So $e = a$	A 1	2.2a
	But this is a contradiction, as the elements e and a are distinct so $ab \neq ba$	A 1	2.4
		(4)	
(ii)(a)	2 has order 4 and 4 has order 2	M 1	1.1b
	7, 8 and 13 have order 4	A 1	1.1b
	11 and 14 have order 2 and 1 has order 1	A 1	1.1b
		(3)	
(ii)(b)	Finds the subgroup $\{1, 2, 4, 8\}$ or the subgroup $\{1, 7, 4, 13\}$	M 1	1.1b
	Finds both and refers to them as cyclic groups, or gives generator 2 and generator 7	A 1	2.4
	Finds $\{1, 4, 11, 14\}$	B 1	2.2a
	States each element has order 2 or refers to it as Klein Group	B 1	2.5
		(4)	
(ii)(c)	J has an element of order 8, (H does not) or J is a cyclic group (H is not) or other valid reason	M 1	2.4
	They are not isomorphic	A 1	2.2a
		(2)	

(13 marks)